8.3 Show that a Quadrilateral is a Parallelogram

Before You identified properties of parallelograms.

Now You will use properties to identify parallelograms.

Why? So you can describe how a music stand works, as in Ex. 32.

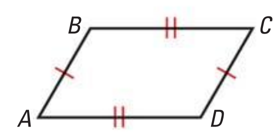
Key Vocabulary
- **parallelogram,** *p. 515*

Given a parallelogram, you can use Theorem 8.3 and Theorem 8.4 to prove statements about the angles and sides of the parallelogram. The converses of Theorem 8.3 and Theorem 8.4 are stated below. You can use these and other theorems in this lesson to prove that a quadrilateral with certain properties is a parallelogram.

THEOREMS — *For Your Notebook*

THEOREM 8.7

If both pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

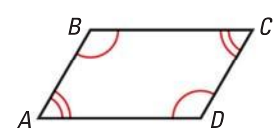


If $\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{AD}$, then $ABCD$ is a parallelogram.

Proof: below

THEOREM 8.8

If both pairs of opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram.



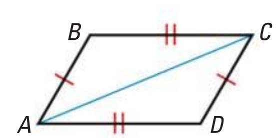
If $\angle A \cong \angle C$ and $\angle B \cong \angle D$, then $ABCD$ is a parallelogram.

Proof: Ex. 38, p. 529

PROOF Theorem 8.7

GIVEN ▶ $\overline{AB} \cong \overline{CD}$, $\overline{BC} \cong \overline{AD}$

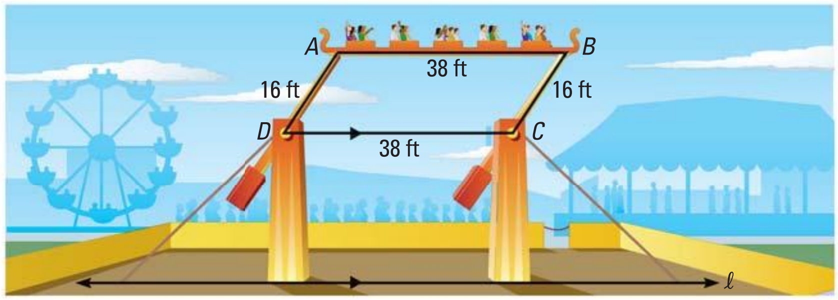
PROVE ▶ $ABCD$ is a parallelogram.



Proof Draw $\overline{AC}$, forming $\triangle ABC$ and $\triangle CDA$. You are given that $\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{AD}$. Also, $\overline{AC} \cong \overline{AC}$ by the Reflexive Property of Congruence. So, $\triangle ABC \cong \triangle CDA$ by the SSS Congruence Postulate. Because corresponding parts of congruent triangles are congruent, $\angle BAC \cong \angle DCA$ and $\angle BCA \cong DAC$. Then, by the Alternate Interior Angles Converse, $\overline{AB} \parallel \overline{CD}$ and $\overline{BC} \parallel \overline{AD}$. By definition, $ABCD$ is a parallelogram.

EXAMPLE 1 Solve a real-world problem

RIDE An amusement park ride has a moving platform attached to four swinging arms. The platform swings back and forth, higher and higher, until it goes over the top and around in a circular motion. In the diagram below, $\overline{AD}$ and $\overline{BC}$ represent two of the swinging arms, and $\overline{DC}$ is parallel to the ground (line ℓ). *Explain* why the moving platform $\overline{AB}$ is always parallel to the ground.



Solution

The shape of quadrilateral *ABCD* changes as the moving platform swings around, but its side lengths do not change. Both pairs of opposite sides are congruent, so *ABCD* is a parallelogram by Theorem 8.7.

By the definition of a parallelogram, $\overline{AB} \parallel \overline{DC}$. Because $\overline{DC}$ is parallel to line ℓ, $\overline{AB}$ is also parallel to line ℓ by the Transitive Property of Parallel Lines. So, the moving platform is parallel to the ground.

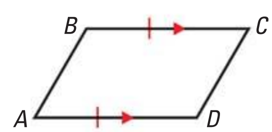
✓ GUIDED PRACTICE for Example 1

1. In quadrilateral *WXYZ*, $m\angle W = 42°$, $m\angle X = 138°$, $m\angle Y = 42°$. Find $m\angle Z$. Is *WXYZ* a parallelogram? *Explain* your reasoning.

THEOREMS *For Your Notebook*

THEOREM 8.9

If one pair of opposite sides of a quadrilateral are congruent and parallel, then the quadrilateral is a parallelogram.

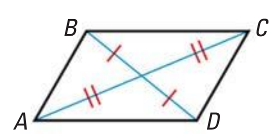


If $\overline{BC} \parallel \overline{AD}$ and $\overline{BC} \cong \overline{AD}$, then *ABCD* is a parallelogram.

Proof: Ex. 33, p. 528

THEOREM 8.10

If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.



If $\overline{BD}$ and $\overline{AC}$ bisect each other, then *ABCD* is a parallelogram.

Proof: Ex. 39, p. 529

EXAMPLE 2 Identify a parallelogram

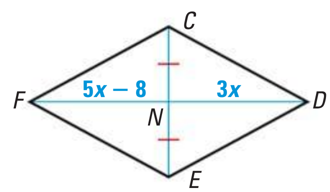
ARCHITECTURE The doorway shown is part of a building in England. Over time, the building has leaned sideways. *Explain* how you know that $SV = TU$.

Solution

In the photograph, $\overline{ST} \parallel \overline{UV}$ and $\overline{ST} \cong \overline{UV}$. By Theorem 8.9, quadrilateral $STUV$ is a parallelogram. By Theorem 8.3, you know that opposite sides of a parallelogram are congruent. So, $SV = TU$.

EXAMPLE 3 Use algebra with parallelograms

ALGEBRA For what value of x is quadrilateral $CDEF$ a parallelogram?



Solution

By Theorem 8.10, if the diagonals of $CDEF$ bisect each other, then it is a parallelogram. You are given that $\overline{CN} \cong \overline{EN}$. Find x so that $\overline{FN} \cong \overline{DN}$.

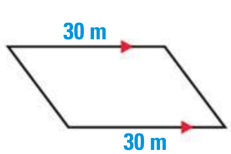
$FN = DN$	Set the segment lengths equal.
$5x - 8 = 3x$	Substitute $5x - 8$ for FN and $3x$ for DN.
$2x - 8 = 0$	Subtract $3x$ from each side.
$2x = 8$	Add 8 to each side.
$x = 4$	Divide each side by 2.

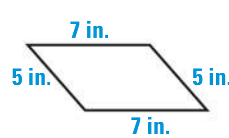
When $x = 4$, $FN = 5(4) - 8 = 12$ and $DN = 3(4) = 12$.

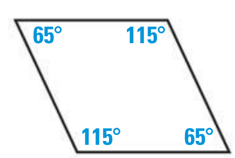
▶ Quadrilateral $CDEF$ is a parallelogram when $x = 4$.

✓ GUIDED PRACTICE for Examples 2 and 3

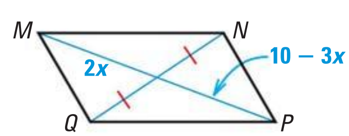
What theorem can you use to show that the quadrilateral is a parallelogram?

2. 

3. 

4. 

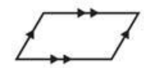
5. For what value of x is quadrilateral $MNPQ$ a parallelogram? *Explain* your reasoning.



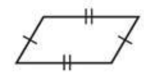
CONCEPT SUMMARY — *For Your Notebook*

Ways to Prove a Quadrilateral is a Parallelogram

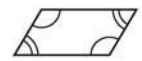
1. Show both pairs of opposite sides are parallel. *(DEFINITION)*



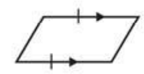
2. Show both pairs of opposite sides are congruent. *(THEOREM 8.7)*



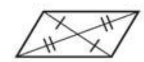
3. Show both pairs of opposite angles are congruent. *(THEOREM 8.8)*



4. Show one pair of opposite sides are congruent and parallel. *(THEOREM 8.9)*



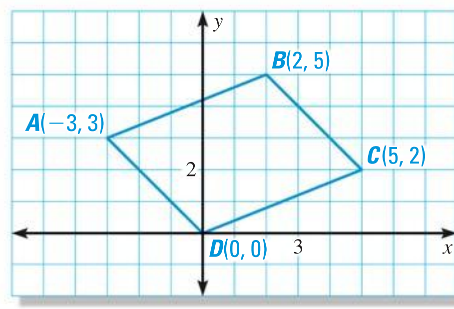
5. Show the diagonals bisect each other. *(THEOREM 8.10)*



EXAMPLE 4 Use coordinate geometry

ANOTHER WAY
For alternative methods for solving the problem in Example 4, turn to page 530 for the **Problem Solving Workshop**.

Show that quadrilateral *ABCD* is a parallelogram.



Solution

One way is to show that a pair of sides are congruent and parallel. Then apply Theorem 8.9.

First use the Distance Formula to show that $\overline{AB}$ and $\overline{CD}$ are congruent.

$$AB = \sqrt{[2-(-3)]^2 + (5-3)^2} = \sqrt{29} \qquad CD = \sqrt{(5-0)^2 + (2-0)^2} = \sqrt{29}$$

Because $AB = CD = \sqrt{29}$, $\overline{AB} \cong \overline{CD}$.

Then use the slope formula to show that $\overline{AB} \parallel \overline{CD}$.

$$\text{Slope of } \overline{AB} = \frac{5-(3)}{2-(-3)} = \frac{2}{5} \qquad \text{Slope of } \overline{CD} = \frac{2-0}{5-0} = \frac{2}{5}$$

Because $\overline{AB}$ and $\overline{CD}$ have the same slope, they are parallel.

▶ $\overline{AB}$ and $\overline{CD}$ are congruent and parallel. So, *ABCD* is a parallelogram by Theorem 8.9.

✓ GUIDED PRACTICE for Example 4

6. Refer to the Concept Summary above. *Explain* how other methods can be used to show that quadrilateral *ABCD* in Example 4 is a parallelogram.

8.3 EXERCISES

HOMEWORK KEY

○ = **WORKED-OUT SOLUTIONS** on p. WS1 for Exs. 5, 11, and 31

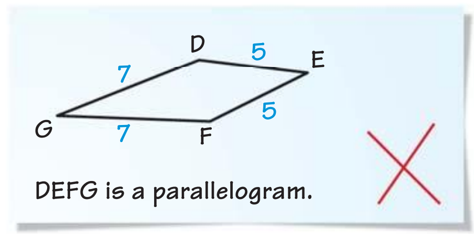
★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 7, 18, and 37

SKILL PRACTICE

1. **VOCABULARY** *Explain* how knowing that $\overline{AB} \parallel \overline{CD}$ and $\overline{AD} \parallel \overline{BC}$ allows you to show that quadrilateral *ABCD* is a parallelogram.

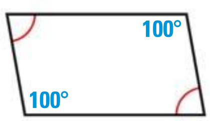
2. ★ **WRITING** A quadrilateral has four congruent sides. Is the quadrilateral a parallelogram? *Justify* your answer.

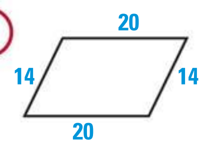
3. **ERROR ANALYSIS** A student claims that because two pairs of sides are congruent, quadrilateral *DEFG* shown at the right is a parallelogram. *Describe* the error that the student is making.

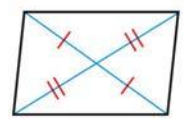


EXAMPLES 1 and 2 on pp. 523–524 for Exs. 4–7

REASONING **What theorem can you use to show that the quadrilateral is a parallelogram?**

4. 

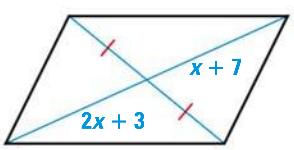
5. 

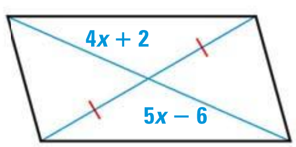
6. 

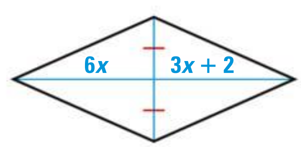
7. ★ **SHORT RESPONSE** When you shift gears on a bicycle, a mechanism called a *derailleur* moves the chain to a new gear. For the derailleur shown below, $JK = 5.5$ cm, $KL = 2$ cm, $ML = 5.5$ cm, and $MJ = 2$ cm. *Explain* why $\overline{JK}$ and $\overline{ML}$ are always parallel as the derailleur moves.

EXAMPLE 3 on p. 524 for Exs. 8–10

ALGEBRA **For what value of *x* is the quadrilateral a parallelogram?**

8. 

9. 

10. 

EXAMPLE 4 on p. 525 for Exs. 11–14

COORDINATE GEOMETRY **The vertices of quadrilateral *ABCD* are given. Draw *ABCD* in a coordinate plane and show that it is a parallelogram.**

11. $A(0, 1), B(4, 4), C(12, 4), D(8, 1)$

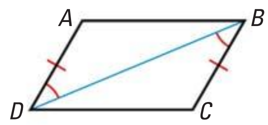
12. $A(-3, 0), B(-3, 4), C(3, -1), D(3, -5)$

13. $A(-2, 3), B(-5, 7), C(3, 6), D(6, 2)$

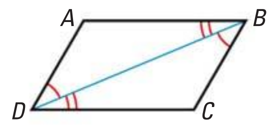
14. $A(-5, 0), B(0, 4), C(3, 0), D(-2, -4)$

REASONING ***Describe*** **how to prove that *ABCD* is a parallelogram.**

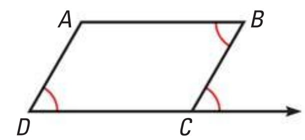
15.



16.



17.



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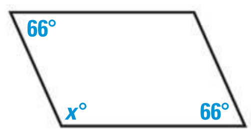
18. ★ **MULTIPLE CHOICE** In quadrilateral $WXYZ$, $\overline{WZ}$ and $\overline{XY}$ are congruent and parallel. Which statement below is not necessarily true?

Ⓐ $m\angle Y + m\angle W = 180°$ Ⓑ $\angle X \cong \angle Z$

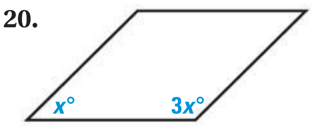
Ⓒ $\overline{WX} \cong \overline{ZY}$ Ⓓ $\overline{WX} \parallel \overline{ZY}$

ALGEBRA **For what value of *x* is the quadrilateral a parallelogram?**

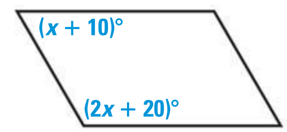
19.



20.



21.

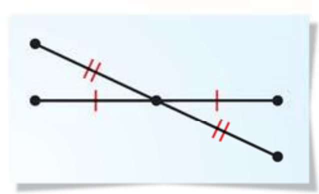


BICONDITIONALS **Write the indicated theorems as a biconditional statement.**

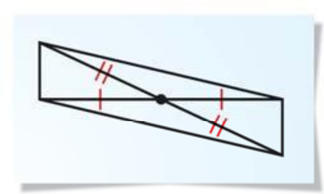
22. Theorem 8.3, page 515 and Theorem 8.7, page 522

23. Theorem 8.4, page 515 and Theorem 8.8, page 522

24. **REASONING** Follow the steps below to draw a parallelogram. *Explain* why this method works. State a theorem to support your answer.



STEP 1 Use a ruler to draw two segments that intersect at their midpoints.



STEP 2 Connect the endpoints of the segments to form a quadrilateral.

COORDINATE GEOMETRY **Three of the vertices of ▱*ABCD* are given. Find the coordinates of point *D*. Show your method.**

25. $A(-2, -3)$, $B(4, -3)$, $C(3, 2)$, $D(x, y)$

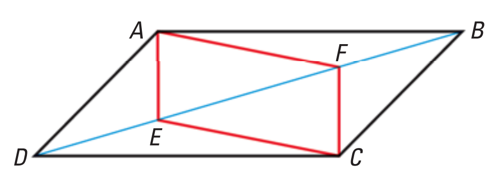
26. $A(-4, 1)$, $B(-1, 5)$, $C(6, 5)$, $D(x, y)$

27. $A(-4, 4)$, $B(4, 6)$, $C(3, -1)$, $D(x, y)$

28. $A(-1, 0)$, $B(0, -4)$, $C(8, -6)$, $D(x, y)$

29. **CONSTRUCTION** There is more than one way to use a compass and a straightedge to construct a parallelogram. *Describe* a method that uses Theorem 8.7 or Theorem 8.9. Then use your method to construct a parallelogram.

30. **CHALLENGE** In the diagram, $ABCD$ is a parallelogram, $BF = DE = 12$, and $CF = 8$. Find AE. *Explain* your reasoning.



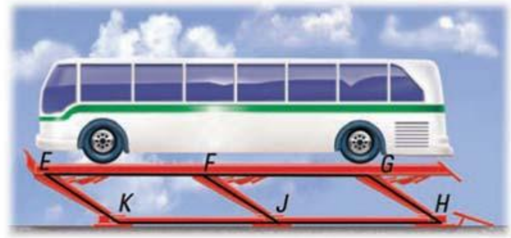
Problem Solving

EXAMPLES 1 and 2
on pp. 523–524
for Exs. 31–32

31. **AUTOMOBILE REPAIR** The diagram shows an automobile lift. A bus drives on to the ramp ($\overline{EG}$). Levers ($\overline{EK}$, $\overline{FJ}$, and $\overline{GH}$) raise the bus. In the diagram, $\overline{EG} \cong \overline{KH}$ and $EK = FJ = GH$. Also, F is the midpoint of $\overline{EG}$, and J is the midpoint of $\overline{KH}$.

a. Identify all the quadrilaterals in the automobile lift. *Explain* how you know that each one is a parallelogram.

b. *Explain* why $\overline{EG}$ is always parallel to $\overline{KH}$.



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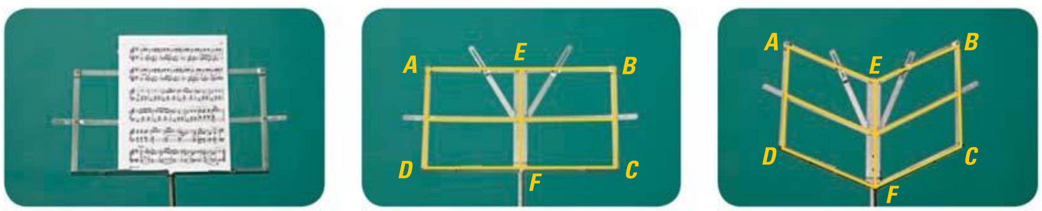
32. **MUSIC STAND** A music stand can be folded up, as shown below. In the diagrams, $\angle A \cong \angle EFD$, $\angle D \cong \angle AEF$, $\angle C \cong \angle BEF$, and $\angle B \cong \angle CFE$. *Explain* why $\overline{AD}$ and $\overline{BC}$ remain parallel as the stand is folded up. Which other labeled segments remain parallel?



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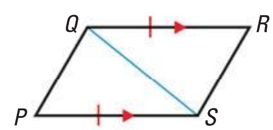
33. **PROVING THEOREM 8.9** Use the diagram of $PQRS$ with the auxiliary line segment drawn. Copy and complete the flow proof of Theorem 8.9.



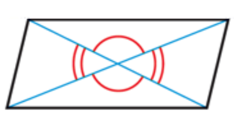
GIVEN ▶ $\overline{QR} \parallel \overline{PS}$, $\overline{QR} \cong \overline{PS}$

PROVE ▶ $PQRS$ is a parallelogram.

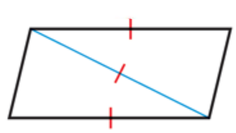
- $\overline{QR} \parallel \overline{PS}$ — Given → $\angle PSQ \cong \angle RQS$ — ?
- $\overline{QS} \cong \overline{QS}$ — ?
- $\overline{QR} \cong \overline{PS}$ — ?
- $\angle PSQ \cong \angle RQS$, $\overline{QS} \cong \overline{QS}$, $\overline{QR} \cong \overline{PS}$ → $\triangle RSQ \cong \triangle PQS$ — ?
- $\triangle RSQ \cong \triangle PQS$ → $\overline{RS} \cong \overline{PQ}$ — ?
- $\overline{RS} \cong \overline{PQ}$, $\overline{QR} \cong \overline{PS}$ → $PQRS$ is a ▱. — ?

REASONING **A student claims incorrectly that the marked information can be used to show that the figure is a parallelogram. Draw a quadrilateral with the marked properties that is clearly *not* a parallelogram. *Explain*.**

34.



35.



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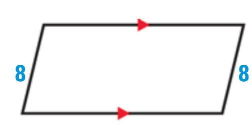
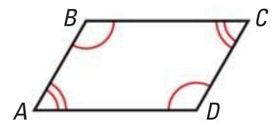


○ = WORKED-OUT SOLUTIONS on p. WS1

★ = STANDARDIZED TEST PRACTICE

37. ★ **EXTENDED RESPONSE** Theorem 8.5 states that if a quadrilateral is a parallelogram, then its consecutive angles are supplementary. Write the converse of Theorem 8.5. Then write a plan for proving the converse of Theorem 8.5. Include a diagram.

38. **PROVING THEOREM 8.8** Prove Theorem 8.8.

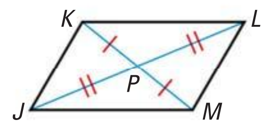


GIVEN ▶ $\angle A \cong \angle C$, $\angle B \cong \angle D$

PROVE ▶ $ABCD$ is a parallelogram.

Hint: Let $x°$ represent $m\angle A$ and $m\angle C$, and let $y°$ represent $m\angle B$ and $m\angle D$. Write and simplify an equation involving x and y.

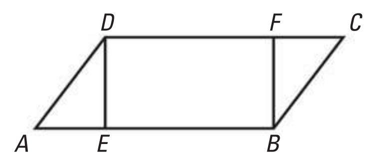
39. **PROVING THEOREM 8.10** Prove Theorem 8.10.



GIVEN ▶ Diagonals $\overline{JL}$ and $\overline{KM}$ bisect each other.

PROVE ▶ $JKLM$ is a parallelogram.

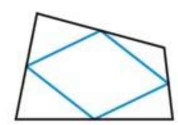
40. **PROOF** Use the diagram at the right.



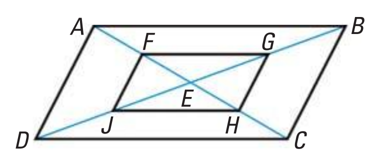
GIVEN ▶ $DEBF$ is a parallelogram, $AE = CF$

PROVE ▶ $ABCD$ is a parallelogram.

41. **REASONING** In the diagram, the midpoints of the sides of a quadrilateral have been joined to form what appears to be a parallelogram. Show that a quadrilateral formed by connecting the midpoints of the sides of any quadrilateral is *always* a parallelogram. (*Hint:* Draw a diagram. Include a diagonal of the larger quadrilateral. Show how two sides of the smaller quadrilateral are related to the diagonal.)



42. **CHALLENGE** Show that if $ABCD$ is a parallelogram with its diagonals intersecting at E, then you can connect the midpoints F, G, H, and J of $\overline{AE}$, $\overline{BE}$, $\overline{CE}$, and $\overline{DE}$, respectively, to form another parallelogram, $FGHJ$.



MIXED REVIEW

PREVIEW
Prepare for Lesson 8.4 in Exs. 43–45.

In Exercises 43–45, draw a figure that fits the description. *(p. 42)*

43. A quadrilateral that is equilateral but not equiangular

44. A quadrilateral that is equiangular but not equilateral

45. A quadrilateral that is concave

46. The width of a rectangle is 4 centimeters less than its length. The perimeter of the rectangle is 42 centimeters. Find its area. *(p. 49)*

47. Find the values of x and y in the triangle shown at the right. Write your answers in simplest radical form. *(p. 457)*